

FRACTION AS A QUANTITY: DESCRIBING STUDENTS' REASONING

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Understanding fraction as a quantity has been identified as a key developmental understanding. In this study, students in Grades 5, 8, and 11 were asked to compare the areas of two halves of the same square—a rectangle and a right triangle. Findings from this study suggest that students who understand fraction as a quantity use reasoning related to a generalization, whereas students who understand fraction as an arrangement use reasoning related to visualization, computation, or characteristics of the specific shapes involved. Knowing the reasoning exhibited by students can inform both teachers and mathematics curriculum writers in the creation of and planning for instructional tasks.

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According to Simon (2006), other researchers (e.g., Skemp's (1976) relational understanding and Hiebert & Lefevre's (1986) conceptual knowledge) have used the term "understanding" in different and productive ways, mainly in outlining valuable types of mathematical knowledge. However, Simon (2006) argued that none have described it in such a way as to "help in identifying critical transitions that are essential for students' mathematical development" (p. 360). To bridge the gap, Simon (2006) introduced a construct known as *key developmental understanding* (KDU). A KDU has two characteristics. The first of which is that a student makes a "conceptual advance" (Simon, 2006, p. 362). Simon (2006) defined "conceptual advance" as "a change in students' ability to think about and/or perceive particular mathematical relationships" (p. 362). The second characteristic of a KDU is that the transition between a student not holding the desired knowledge and holding it requires an accumulation of activity and reflection over the course of multiple experiences. In other words, the desired knowledge cannot be obtained purely through explanation or demonstration.

To further illustrate the construct of a KDU, Simon (2006) provided an episode from a fourth-grade classroom. Students were given geoboards and asked to use a blue rubber band to make a square and a red rubber band around half of their square. Simon (2006) stated that most students demonstrated half, with the red rubber band, by making two congruent rectangles. However, one student showed half by splitting the square into two congruent right triangles. The students' consensus was that both representations of half were correct. The justification involved reasoning that "each of the parts was one of two equal parts and that the two parts made up the whole" (p. 361). When Simon asked students to compare the rectangle half and the right triangle half, their responses were split between three comparisons: the rectangle was larger, the right triangle was larger, and the two shapes were the same size.

Simon (2006) asserted that the students who believed that the rectangular or triangular half was larger viewed a fraction, one-half in this case, as an arrangement. In other words, the whole was separated into identical parts. Students who believed that the rectangle and right triangle were the same size viewed fraction as a quantity. These students demonstrated a conceptual advance, or KDU, in recognizing that the shapes were congruent and that "equal partitioning creates specific units of quantity" (p. 362). According to Simon (2006), researchers must observe

students participating in the same mathematical task and examine the differences in their actions and responses to identify or better understand a KDU. The geoboard task was utilized as a context to highlight an important aspect in understanding fractions.

Similar contexts have been used in other research studies, although with different goals. In Simon et al. (2004), researchers posed a similar task, in which the original squares were cookies, was posed to a third-grade student. The student, Micki, thought that the right triangle was larger than the rectangle. This example was used to exemplify the progression of students' learning processes. In Simon et al. (2018), three fourth-grade participants were given a similar task as part of a pre-assessment, in which the original shapes were rectangular brownies. The pre-assessment results informed an instructional intervention with a goal to promote fraction as measure understanding. None of the students recognized that the rectangular brownie and right triangular brownie were the same size even though they were half of the same whole brownie. Instead, they chose the right triangular piece, claiming it was larger than the rectangular piece. Similarly, in a longitudinal study by Barrett et al. (2017), a similar task was presented to students to understand area conservation. Researchers observed significant inconsistency in student responses, which prompted revisions to a developmental progression involving measurement. However, the reasoning students used to compare shapes in these tasks was not discussed or analyzed in depth. In Barrett et al. (2017), fourth-grade students were given pairs of pre-cut paper shapes in order to compare areas: a square and a right triangle formed by cutting a congruent square along the diagonal, the same square and a rectangle formed by cutting a congruent square in half vertically, and the right triangle and the rectangle, both made from cutting the square. Participants' responses indicated that some students believed the rectangle had a larger area, while others stated that the triangle had a larger area.

Within the contexts described in the current literature (i.e., Barrett et al., 2017; Simon et al., 2004; Simon, 2006; Simon et al., 2018), detailed information about students' reasoning, as it relates to their understanding of fractions (as an arrangement versus as a quantity) has not been communicated. This study describes the reasoning of students as they compared areas of squares, rectangles, and right triangles, and in doing so, extends the existing literature by examining students' reasoning related to each type of understanding, fraction as an arrangement to reasoning about fraction as a quantity, and the transition between.

Research Questions

In this study, we investigated fifth-, eighth-, and eleventh-grade students' reasoning as it related to comparing the areas of rectangles and right triangles. The two research questions addressed were:

1. What reasoning do students use to compare the areas of squares, rectangles, and right triangles?
2. What does it look like for a student to demonstrate understanding of fraction as an arrangement when comparing areas of squares, rectangles, and right triangles? Fraction as a quantity?

Theoretical Perspective

Constructivism is the theoretical perspective that informs this study. Constructivism is based on the idea that learners build their own knowledge, understanding, and ways of reasoning over time by interacting with their environment (von Glaserfeld, 1995). In other words, knowledge cannot be merely transferred, by explanation or demonstration, to the student. Students'

accumulation of experiences, reflection, and abstraction provides structure for developing their conceptual understanding. In addition, the role of the researcher or teacher is to listen to students and make sense of their conceptual understanding. Listening to students is what leads to a continually updated hypothesis or anticipation of student learning. Then, that interpretation can facilitate the way in which misconceptions are addressed and the type of intervention to attempt to further develop students' understandings (von Glaserfeld, 1995).

Methods

As part of a larger study investigating how students reasoned about areas of triangles, the research team conducted semi-structured individual interviews with fifth-, eighth-, and eleventh-grade students during the summer of 2020. Participants were recruited from a public school district in the Midwestern region of the United States. The district is located in a suburban community and has 1,000 students in attendance of which 74.1% identified as White, 4% as Black, 7.9% as Hispanic, 7.2% as Asian, and 6.8% as multi-racial. In this paper, we examine participants' responses to the first task of the first interview.

Procedure

Participants completed at least one zoom-recorded interview. Considering the interviews were conducted through zoom, participants were in places where other people could have been present; however, researchers asked that they not interact with the participant during the interview. All materials associated with the interview (i.e., ruler, blank paper, printed tasks) were mailed to or dropped off at the residences of the participants.

The first interview lasted on average 20 minutes. During each interview, two of the three researchers were present. One researcher acted as the main interviewer, posing each of the tasks and asking follow-up questions, while the other researcher was mainly an observer. At times, something the participant said may have been uncertain and the second researcher asked further questions.

We posed this task, consisting of four parts, to 22 participants: 14 students in Grade 5, four students in Grade 8, and four students in Grade 11. For task 1a, the participant was shown a yellow paper square as the interviewer asked, "If I asked you what the area of this square was, what would I mean by area?" If the participant's response focused on multiplying sides of the square, then the interviewer asked, "What if you were talking to a second grader that does not know about multiplication? How would you describe area?" For task 1b, the interviewer showed the original, yellow paper square along with a red paper square of the same size. The interviewer folded the red paper square in half, vertically, making a rectangle. The participant was asked, "Compare the area of these two figures. How do you know?" For task 1c, the participant was shown the original, yellow paper square and a blue paper square of the same size. The blue paper square was folded in half, from corner to opposite corner, making a right triangle. Again, the interviewer asked, "Compare the area of these two figures. How do you know?" For the last part of Task 1, the participant was shown the rectangle from 1b and the right triangle from 1c and asked, "Compare the area of these two figures. How do you know?"



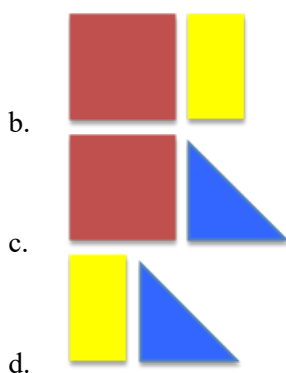


Figure 1: Images of Folded Shapes for Task 1

Data Analysis

After all interviews were completed, the team of three researchers worked through three stages of data analysis. During the first stage, the interviews were divided among the three researchers. For each interview, the assigned researcher watched it and created a written summary. Each summary consisted of a loose transcription of the interview along with words or phrases that described participants' ways of reasoning. These words and phrases became codes that the researchers discussed, sorted into groups, and categorized into official codes. The updated codes were utilized when each interview and its summary was revisited again by the two other researchers who had not created it originally.

During the second stage of data analysis, all three researchers independently coded each task and then met to discuss those codes. For the final stage, pairs of researchers independently coded and then, again, all three researchers met to discuss. Interrater reliability was tracked for each interview. Scores ranged from 39 to 91%, with an average of 68%.

For the final stage of data analysis, pairs of researchers independently coded tasks and then together, compared codes. Interrater reliability scores ranged from 71 to 100%, with an average of 86%. If any discrepancies arose throughout this process, all researchers met to discuss and resolve them by revising or clarifying codes.

Findings

Overall, we identified two main categories to distinguish students' reasoning about the areas of the rectangle and right triangle. Sixteen participants stated that the rectangle and right triangle had equal areas, and six participants stated that the rectangle and right triangle had unequal areas. However, within those two categories, we identified participants who wavered in their decision. This caused us to create subcategories. We describe below the nuances we found in students' reasoning within each subcategory.

Table 1: Participants' Reasoning Categorization

Participants who stated that rectangle and right triangle had equal areas	501*, 504, 505, 506, 508, 509*, 511*, 512*, 513, 514
	802, 803*
	1101, 1103, 1104, 1105
Participants who stated that the rectangle and right triangle had unequal areas	502, 503, 507, 510*
	801*, 804

*Participants who stated that the shapes' areas were both equal and unequal during the interview. The number of hundreds indicates the grade level. For example, participant 1101 was in Grade 11 at the time of the study.

Students' Reasoning Related to Both Shapes Having Equal Areas

While 16 participants decided that the rectangle and right triangle had equal areas, 11 of them (504, 505, 506, 508, 513, 514, 802, 1101, 1103, 1104, 1105) promptly stated their reasoning. One participant (508) reasoned that the rectangle and right triangle are "equal because they were both half of the square." This response, with an emphasis on the relationship between each shape and the original square, was recorded by seven other participants (508, 513, 514, 1101, 1103, 1104, 1105). Four participants referred to the action of folding the paper square while comparing the areas of the halves. For example, one participant (504) stated "they're both still folded in half and they're both still the same size as the square if they're unfolded." Whereas three participants (505, 506, 802) used reasoning that both halves were the same because they had been folded from the square in different directions. For example, 505 stated that the rectangle and right triangle were "just folded in half two different ways."

Seven participants (501, 509, 510, 511, 512, 801, 803) expressed some hesitation when asked to compare the areas. In other words, they provided at least one reason each for the rectangle and right triangle to have both equal and unequal areas. Five participants (501, 509, 511, 512, 803) switched their reasoning from the rectangle and right triangle having equal areas to unequal areas during the interview, but in the end, stated that the halves had equal areas. Except for 803, participants switched back and forth between claims that shapes have equal areas or unequal areas, at least twice. Interestingly, within a matter of less than 4 minutes, 510 switched reasoning five times.

Of these five participants, two (511, 803) utilized numerical measurements to make a final decision about how the areas of the rectangle and right triangle compared. 511 began by stating that the areas of the shapes were the same because "they're both half of the square." Then, while 511 described the computational process of finding the area of the right triangle, "you multiply length and width to get the triangle, but divide it in half, because it's a triangle", 511 stopped and said "But the rectangle... I guess it wouldn't be then." 511 proceeded to then switch back to state the areas were different because "the triangle's sides are bigger." When the interviewer asked, "What are you grappling with?", 511 said, "I guess I just don't know the numbers... the length of the square... the width of it." The interviewer then asked 511 to describe the measurements she wanted. From the square, 511 stated that she wanted "the length of one side of it and the width of one side of it." 511 also asked for "the width" or "the bottom of the rectangle." After the interviewer supplied these measurements, 511 computed the area of the square, the rectangle, and the triangle. When 511 determined that the rectangle and triangle areas were 6 and 6, 511 stated that the areas would be equal. 803's interview began similarly with 803 stating that the areas were the same because "you folded it in half and since they're a square, they have equal portions." However, continuing that same statement, 803 said "but I don't think they are the same because you didn't fold a line, you folded a diagonal so the triangle would be bigger." The interviewer responded by asking about these conflicting conclusions, "What made you think they might have been the same at first?" 803 restated both ideas, the first being the areas are equal because the shapes are both half of the same square, and the second being that the triangle's area is greater because it was made by folding on a diagonal. Then, the interviewer asked, "Which one are you most convinced by?" to which 803 explained that she saw "both points" but was

more convinced by the areas being equal because she assigned values to the square and computed the areas of the rectangle and right triangle to realize they were the same.

The other three participants (501, 509, 512) who went back and forth but ultimately decided the areas were indeed equal used reasoning related to both shapes being half of the original square. Both 501 and 509 were only swayed once when deciding if the two areas were equal. At first, 501 was conflicted when it came to finding the area of the triangle, stating “it’d be hard because the triangle has points that aren’t the same as the square because it’s cut off.” Initially, 509 reasoned that the triangle’s area could be larger because “the triangle looks a little bigger than the rectangle.” 512 began by stating that since both shapes were half of the original square that their areas “might be equal.” The interviewer asked the participant to clarify his hesitance and 512 explained that the shape of the triangle made him doubt if its area was larger, “it looked like it could have had more area because it is more spread out.”

Students' Reasoning Related to Both Shapes Having Unequal Areas

Six participants decided that the areas of the triangle had a larger area than the rectangle. However, four participants (502, 503, 507, 804) promptly stated that the rectangle and right triangle had unequal areas. At no point in the interview did these participants explicitly conclude that the areas of the rectangle and right triangle were equal, differentiating them from the reasoning exhibited in all other interviews. These participants exhibited diverse ways of claiming that the area of the right triangle was larger.

One participant (502) came to a swift conclusion stating that the area of triangle was bigger than the rectangle “from the way it was folded.” Two participants (503 and 507) argued that the right triangle had a larger area because it has a longer side than the rectangle. For the previous three parts of this task, both 503 and 507 focused their responses and comparisons on the computation involved in finding the areas. 503 continued this type of reasoning into the final part. For instance, 503 stated “I know the triangle is only half of the square, but then the triangle has a longer side which when you multiply it, it would get a bigger number than multiplying two of the same sides of the square that would not get a bigger number.” This computational way of reasoning about products of lengths also led 503 to make the claim that the triangle might have a larger area than the original square even after stating he knew the triangle “is only half the size of” the original square. 507 also attempted to reason about products. 507 first stated that the rectangle and right triangle were “both half the size of the square, so it just might be different of how you’re gonna get the area and the numbers might be a little bit different.” To explain how he knew that the areas of the two shapes were different, 508 asserted that the triangle “has one right angle and it’s got one longer side than the rectangle does.”

Another participant, 804, used reasoning related to an action to decide that the areas were different. When asked to compare the two shapes’ areas, 804 explained that “The triangle seems bigger. I think if I put the rectangle inside the triangle and cut it to like make it fit, the triangle would have more area.”

In contrast to 502, 503, 507, 804, when two participants (510, 801) were asked to compare the areas of the rectangle and right triangle, they went back and forth with their responses. However, they ultimately decided that the halves had different or unequal areas. In one interview, 510 presented several back-and-forth decisions as the interviewer asked for clarification. First, 510 stated that the right triangle was bigger because “it looks like it has more space than the rectangle.” When asked by the interviewer, “How do you know?”, 510 stated that the right triangle and rectangle were equal in area “because they both cover half the square evenly.” The interviewer repeated the participant’s two conflicting thoughts and asked, “Which

one are you most convinced by?” 510 responded that the two shapes had different areas and that the “triangle has a little more space than the rectangle.” The interviewer then asked, “So, you think the triangle looks like it has more space?”, to which 510 responded, “I kind of feel like it’s the rectangle, but I’m not sure.” Responding to another switch in reasoning, the interviewer asked, “What’s making you not confident in one or the other?”, 510 then described how “if you put the rectangle up to the triangle, there’s a little more space, a little triangle... but when you flip it over the other way, the rectangle takes over the whole thing.” Finally, the interviewer restated the prompt, “So if we compare the area of the triangle and the rectangle, do they have the same area or do they have different areas?”, 510 stated that “they have different areas” and that the right triangle has a larger area.

While the interview with 801 had fewer back and forth moments, 801’s final decision was also that the halves had unequal areas. However, 801 seemed to use information about the relationship between each shape and the original square, mainly the formulas to find the area of each shape. For instance, 801 stated that “the triangle has an area that is exactly half of the square” and since the rectangle “is already half the width, but the same length, so it’s already half the area.” When the interviewer asked, “so, are the areas different or the same?”, 801 said that the areas were different “because there are different ways to find the areas.” 801 went on to state that when you “visually compare them, the triangle seems to cover more area of the square.”

Discussion

The data collected and analyzed from this task provided further evidence of what Simon claimed is a KDU: fraction as a quantity. As Simon (2006) expressed, students who responded that the rectangle and the right triangle were different, even after stating each were half of the square when presented separately, viewed fraction (half) as an arrangement. The students who stated that the rectangle and the right triangle were the same viewed fraction (half) as a quantity. However, the reasons for which a student believed each position to be true is what sets this study apart from what has been reported thus far.

Because Simon (2006) stated that an accumulation of experiences and activities was essential to a KDU, we were not surprised that the all four eleventh-grade students asserted that the rectangle and the right triangle were equal in area because they were both half of the same shape. This reasoning was exhibited by most students who believed that the rectangle and the right triangle had equal areas. Considering this study incorporated the action of folding of the squares into the rectangle and the right triangle, students also used reasoning about both folding and unfolding of the shapes. For instance, 504 stated “they’re both still folded in half and they’re both still the same size as the square if they’re unfolded,” and 505 stated that the halves are “just folded in half two different ways.” These two participants used folding and unfolding as a reason for the shapes to have equal areas. However, 502 used it as a reason for the right triangle to have a larger area “from the way it was folded.” The responses discussed so far were supported by generalizable knowledge. In other words, the students did not need to refer to the specific shapes but spoke in rules (e.g., “they are both half the same shape”). However, two fifth-grade students were more convinced to conclude that the areas of the two shapes were equal when they knew specific measurements and could calculate the areas of each.

Regarding the reasoning observed by the students who stated that the rectangle and the right triangle having unequal areas, there was also variety. Reasons revolved around visualization (i.e., superimposing shapes, folding, or cutting), computation, or using characteristics of the specific shapes. For example, 801 stated that when you “visually compare them, the triangle seems to

cover more area of the square,” 507 reasoned that the triangle “has one right angle and it’s got one longer side than the rectangle does.” As mentioned above, only one student, 502, referred to the action of folding, but in this case, to argue the opposite conclusion from the students above. Another student, 801, referred to the area formulas for each shape when she stated the triangle is bigger “because there are different ways to find the areas.” All these reasons either rely on seeing the specific shapes involved, analyzing characteristics of the shape, separate from the other, or anticipating actions being done to the specific shape.

Limitations and Directions for Future Research

Due to the COVID-19 pandemic, all interviews were conducted over zoom. While this medium offered a way to continue to interview students when other formats were not available, it was difficult at times to clearly show the action of folding shapes and interpret student responses. For instance, when students offered reasoning related to turning or superimposing the shapes, the interviewer had to interpret the verbal directions the students offered. Future studies should conduct interviews in-person in which the student can hold and manipulate the shapes themselves to express their thoughts. Another limitation of this study was that it included a small sample of students, all recruited from the same school. Future studies may include a larger, more diverse population of students to interview.

Conclusion and Implications

Simon (2006) claimed that it is essential to not only identify KDUs, such as fraction as a quantity, but also to describe the differences in the sophistication of students’ responses in elaboration of a KDU. In this study, we extended Simon’s work on KDUs to examine students’ reasoning about areas of squares, rectangles, and triangles. We noticed that there was variety in students’ comparisons of half as it related to areas of shapes. Specifically, even though students demonstrated that they viewed fraction as a quantity by stating that two halves, that appeared different, were the same, their reasons were different. Some reasoned about the action of folding and being a part of the same original shape and some reasoned about each shape’s numerical measurements. According to Simon (2006), identification of KDUs “is a way of specifying developmental steps” (p. 367) and further elaboration makes the steps in between more apparent. Additionally, participants’ reasoning, particularly the explanations in which participants went back and forth as they compared shapes, was significant and more thoroughly depicted than research that has been conducted thus far using similar tasks.

Both teachers and mathematics curriculum writers would benefit from knowing more about how students think about a fraction, either as an arrangement or as a quantity. According to Simon (2006), a KDU is not usually something that a student can learn solely from a teacher’s or peer’s explanation or demonstration or viewed from the perspective of the adult. These characteristics can make it challenging to develop instructional tasks that advance students’ understanding. These findings could influence a sequence of instruction involving measurement concepts. Future research may include teaching experiments to identify details related to specific tasks or interventions used to progress students’ reasoning from using characteristics of specific shapes toward generalization.

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